

Drinfeld basis for string-inspired Baxter operators

Andrew Rolph and Alessandro Torrielli ^{*}

*Department of Mathematics, University of Surrey
Guildford GU2 7XH, United Kingdom*

Abstract

We propose Drinfeld's second realisation of the quantum group relevant to the Lax-operator approach developed in the work of Bazhanov, Frassek, Lukowski, Meneghelli and Staudacher.

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^{*}*E-mail addresses:* andrew.d.rolph@gmail.com, a.torrielli@surrey.ac.uk

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1. Introduction

In recent years, tremendous progress has been made towards a complete solution of the AdS/CFT integrable system [1], with the two sides of the correspondence matching to a dazzling degree of sophistication and accuracy. Nevertheless, there remains a feeling that a full mathematical understanding of the mechanism may require additional investigation. One route to gather more data which is currently being pursued is the exploration of similar integrable structures in lower-dimensional instances of the correspondence [2].

In [3], on the other hand, a program was started aimed at investigating the algebraic properties of the original AdS₅ system by means of the Baxter Q -operator approach. This program began with a revisitation of the spin- $\frac{1}{2}$ Heisenberg spin-chain in the light of Q -operators, where interesting ‘elementary’ (or *partonic*) Lax operators were employed, and their algebraic relations and representation theory studied. These objects were then related to the theory of Yangians, hence it is natural to ask whether one might develop an analogue of Drinfeld’s second realisation [4] for their description.

In this brief note, we propose an answer to this question. We derive a set of defining relations for the analogue of the Drinfeld generators of ordinary Yangians, by mimicking the procedure of triangular factorisation that works in the standard case. We obtain the relevant oscillator representation which makes contact with [3], and construct the Hopf algebra maps which turn our structure into a quantum group. The resulting formulae bear resemblance to the ordinary Yangian, but also present some more uncommon features which make for an interesting algebraic object.

We should mention work that relates to ours, and which represents an important concurrent line of investigation in this direction. The partonic Lax operator originally appeared in the literature in connection with the so-called “Discrete Self-Trapping” (DST) chain [5]. The reader is referred to [6] for further details¹. Furthermore, in the work of Chicherin,

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Derkachov, Karakhanyan and Kirschner [7] general solutions for the Baxter operators relevant to our discussion are obtained, and similar factorisation properties to the ones we will use here are employed.

2. The $\mathfrak{sl}(2)$ quantum group

2.1. The standard case

A standard and all-important object in the theory of Yangians is represented by the following rational R-matrix:

$$R(u) = u \mathbb{1} + \mathcal{P}, \quad (2.1)$$

with \mathcal{P} the permutation operator. Starting from the R-matrix restricted to the $\mathfrak{sl}(2)$ Lie algebra, one considers the Lax operator

$$L(z) = \begin{pmatrix} z+h & f \\ e & z-h \end{pmatrix}, \quad (2.2)$$

with the generators appearing satisfying the $\mathfrak{sl}(2)$ commutation relations

$$[h, e] = e, \quad [h, f] = -f, \quad [e, f] = 2h. \quad (2.3)$$

One can check that this is equivalent to

$$R(u-v) [L(u) \otimes \mathbb{1}] [\mathbb{1} \otimes L(v)] = [\mathbb{1} \otimes L(v)] [L(u) \otimes \mathbb{1}] R(u-v), \quad (2.4)$$

or, in other words, to

$$(u-v) [L_{ij}(u), L_{hk}(v)] = L_{hj}(u) L_{ik}(v) - L_{hj}(v) L_{ik}(u), \quad (2.5)$$

for $i, j, h, k = 1, 2$. This provides the starting point for the quantum inverse scattering method [8].

2.2. Gauss decomposition

The next step towards Drinfeld's second realisation of the ordinary Yangian is to apply the following *Gauss decomposition* [9–11]:

$$L(z) = \begin{pmatrix} 1 & 0 \\ F(z) & 1 \end{pmatrix} \begin{pmatrix} D_1(z) & 0 \\ 0 & D_2(z) \end{pmatrix} \begin{pmatrix} 1 & E(z) \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} D_1 & D_1 E \\ F D_1 & F D_1 E + D_2 \end{pmatrix}. \quad (2.6)$$

One can determine the commutation relations for the *currents* $D_i(z)$, $E(z)$, $F(z)$ by substituting the above decomposition (2.6) into (2.5), or equivalently into (2.4) using (2.1). This gives:

$$\begin{aligned}
[D_i(u), D_j(v)] &= 0, \quad i, j = 1, 2 \\
(u-v) [D_1(u), E(v)] &= D_1(u)[E(v) - E(u)], \\
(u-v) [D_1(u), F(v)] &= [F(u) - F(v)]D_1(u), \\
(u-v) [D_2(u), E(v)] &= D_2(u)[E(u) - E(v)], \\
(u-v) [D_2(u), F(v)] &= [F(v) - F(u)]D_2(u), \\
(u-v) [E(u), E(v)] &= [E(u) - E(v)]^2, \\
(u-v) [F(u), F(v)] &= -[F(u) - F(v)]^2, \\
(u-v) [E(u), F(v)] &= D_1^{-1}(u)D_2(u) - D_1^{-1}(v)D_2(v).
\end{aligned} \tag{2.7}$$

Drinfeld's second realisation of the ordinary Yangian is at this point typically achieved by expanding the above currents in *modes* (see below), and deducing how the relations (2.7) translate to relations on the modes of the expansion. We will illustrate this procedure in the next subsection for the novel case discussed in this paper.

2.3. The novel case

The authors of [3] consider instead an alternative (*partonic*) Lax operator:

$$L(z) = \begin{pmatrix} z + h & a^\dagger \\ a & 1 \end{pmatrix}, \tag{2.8}$$

with

$$[a, a^\dagger] = 1, \quad h = a^\dagger a. \tag{2.9}$$

One can check that (2.4) still holds.

Our idea is to adopt the very same decomposition as in (2.6) for this new Lax operator. When we then come to expressing the currents in terms of modes, in order to obtain a natural identification with the form of (2.8), we find it necessary to impose a slightly different expansion than the one used for standard Yangians in [10]. We employ the following mode-expansion²:

$$E(u) = \sum_{k=0}^{\infty} \xi_k^+ u^{-k-1}, \quad F(u) = \sum_{k=0}^{\infty} \xi_k^- u^{-k-1}, \quad \Gamma(u) = 1 - \sum_{k=0}^{\infty} \kappa_k u^{-k-1}, \tag{2.10}$$

with

$$\Gamma(u) = 1 + D_1^{-1}(u)D_2(u). \tag{2.11}$$

²The change with respect to [10] is only in the Cartan part $\Gamma(u)$.

This implies that (2.7) is equivalent to the following system of defining relations³:

$$\begin{aligned}
[\kappa_n, \kappa_m] &= 0, & [\kappa_0, \xi_m^\pm] &= 0, \\
[\xi_m^+, \xi_n^-] &= \kappa_{m+n}, \\
[\kappa_m, \xi_{n+1}^\pm] - [\kappa_{m+1}, \xi_n^\pm] &= \pm\{\kappa_m, \xi_n^\pm\}, \\
[\xi_m^\pm, \xi_{n+1}^\pm] - [\xi_{m+1}^\pm, \xi_n^\pm] &= \pm\{\xi_m^\pm, \xi_n^\pm\}.
\end{aligned} \tag{2.12}$$

These relations are different from those of the standard Yangian [4, 10, 11]. The crucial distinction at this stage seems to be the central element κ_0 and the relations involving it. We will see in the next section that more differences are to be observed at the co-algebra level, when we will define a coproduct compatible with (2.12).

The representation one obtains from the explicit use of (2.8) is as follows:

$$\xi_m^+ = (-h)^m a^\dagger, \quad \xi_m^- = a(-h)^m, \quad \kappa_m = (-)^m [h^{m+1} - (h+1)^{m+1}]. \tag{2.13}$$

Let us make here an interesting observation⁴. Had we exchanged the rows and columns of (2.8) while adopting the same decomposition (2.6), we would have obtained the following:

$$D_1(u) = 1, \quad E(u) = a, \quad F(u) = a^\dagger, \quad D_2(u) = u. \tag{2.14}$$

By construction, this still satisfies the whole set of relations (2.7) - most of which now reduces to $0 = 0$ identities. However, one can observe from (2.14) that such an assignment extracts from our new quantum group (2.12) the “level-zero” subalgebra only, *i.e.* the Heisenberg algebra (2.9).

To obtain an *evaluation* representation depending on a spectral parameter λ one just needs to substitute $h \rightarrow h + \lambda$ in ξ_m^\pm in (2.13) (and re-calculate κ_m , for instance as $[\xi_0^+, \xi_m^-]$). In fact, the quantum group we have found is invariant under the *shift automorphism*

$$\kappa_1 \longrightarrow \kappa_1 + \mu \kappa_0, \quad \xi_1^\pm \longrightarrow \xi_1^\pm + \mu \xi_0^\pm, \tag{2.15}$$

with a complex parameter μ .

One thing to notice is that there is no obvious way to “mechanically” generate all higher levels given the level 0 and 1 generators, as κ_0 is central. This is at odds with the case of the standard Yangian, where such a mechanical procedure can be found [12]. It also means that we should pay particular attention to all the statements that extend beyond level 1.

We can perform a map to the analogue of Drinfeld’s first realisation of our quantum group. Let us define

$$\widehat{\kappa} \equiv \frac{\kappa_1}{2} - \xi_0^- \xi_0^+, \quad \widehat{\xi}^+ \equiv \xi_1^+ - \frac{1}{2} \kappa_0^{-1} \kappa_1 \xi_0^+, \quad \widehat{\xi}^- \equiv \xi_1^- - \frac{1}{2} \kappa_0^{-1} \xi_0^- \kappa_1, \tag{2.16}$$

³A similar construction for the Lax operator (2.2) would produce the defining relations of the Yangian in Drinfeld’s second realisation [4, 9, 10].

⁴We thank the referee for pointing out this possibility to us.

where we have allowed ourselves to take inverse powers of the central element κ_0 . One can show that the new generators satisfy $[T^a, \widehat{T}^b] = f_c^{ab} \widehat{T}^c$, namely

$$[\kappa_0, \widehat{\cdot}] = 0 = [\xi_0^\pm, \widehat{\kappa}] \quad [\xi^\mp, \widehat{\xi}_0^\pm] = \mp \widehat{\kappa}, \quad (2.17)$$

where $\widehat{\cdot}$ denotes any of the generators (2.16).

2.4. Coproduct

The coproduct we can equip our algebra with is rather uncommon. It can be obtained from the familiar formula

$$\Delta(L_{ij}(u)) = L_{ik}(u) \otimes L_{kj}(u), \quad (2.18)$$

however the expansion in powers of u reserves a few surprises with respect to the analogous computation performed for the ordinary Yangian. Expanding as in (2.10), and with a few manipulations, we obtain, for example for the first two levels, the unusual

$$\begin{aligned} \Delta(\xi_0^+) &= 1 \otimes \xi_0^+, & \Delta(\xi_1^+) &= 1 \otimes \xi_1^+ + \xi_0^+ \otimes \kappa_0, \\ \Delta(\xi_0^-) &= \xi_0^- \otimes 1, & \Delta(\xi_1^-) &= \xi_1^- \otimes 1 + \kappa_0 \otimes \xi_0^-, \\ \Delta(\kappa_0) &= 0, & \Delta(\kappa_1) &= \kappa_0 \otimes \kappa_0. \end{aligned} \quad (2.19)$$

One can check that this satisfies (2.12), providing a homomorphism of the set of defining relations.

The coproduct (2.19) is rather dissimilar from the one typically assigned to the standard Yangian [4, 10, 11]. The main difference is in the “level-zero” comultiplication rule, which is surprisingly *a-symmetric*. Where one would have expected a formula of the type $\Delta(\mathfrak{t}_0) = \mathfrak{t}_0 \otimes 1 + 1 \otimes \mathfrak{t}_0$, one sees instead a quite peculiar “halved” coproduct in (2.19)⁵. We believe this to be the most significant piece of contradistinction of our new quantum group with respect to the ordinary Yangian.

3. Conclusions

In this short letter, we have proposed Drinfeld’s second realisation for the Baxter-operator approach of [3]. This realisation is traditionally best suited to derive important algebraic quantities, such as the universal R-matrix, and to study representations of the ordinary Yangian. We believe therefore that the realisation we have derived might play the same role in the present case, and favour progress in the program inaugurated by the authors of [3].

We have found that the defining relations and Hopf-algebra coproduct which characterise our quantum group are somewhat similar to the standard ones, however they present some crucial differences which stimulate curiosity in this novel algebraic structure. We plan to come back to the full investigation of these aspects in future work.

⁵One could almost attempt to say that the partonic Lax operator of [3] produces a *partonic* coproduct, meaning that the comultiplication map itself gets also decomposed into some more elementary building blocks (its two “halves”).

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References

- [1] N. Beisert, C. Ahn, L. F. Alday, Z. Bajnok, J. M. Drummond, L. Freyhult, N. Gromov and R. A. Janik *et al.*, “Review of AdS/CFT Integrability: An Overview,” *Lett. Math. Phys.* **99** (2012) 3 [arXiv:1012.3982 [hep-th]] • G. Arutyunov and S. Frolov, “Foundations of the $AdS_5 \times S^5$ Superstring. Part I,” *J. Phys. A* **42** (2009) 254003 [arXiv:0901.4937 [hep-th]].
- [2] A. Sfondrini, “Towards integrability for AdS_3/CFT_2 ,” arXiv:1406.2971 [hep-th] • B. Hoare, A. Pittelli and A. Torrielli, “Integrable S-matrices, massive and massless modes and the $AdS_2 \times S^2$ superstring,” *JHEP* **1411** (2014) 051 [arXiv:1407.0303 [hep-th]].
- [3] V. V. Bazhanov, T. Lukowski, C. Meneghelli and M. Staudacher, “A Shortcut to the Q -Operator,” *J. Stat. Mech.* **1011** (2010) P11002 [arXiv:1005.3261 [hep-th]] • V. V. Bazhanov, R. Frassek, T. Lukowski, C. Meneghelli and M. Staudacher, “Baxter Q -Operators and Representations of Yangians,” *Nucl. Phys. B* **850** (2011) 148 [arXiv:1010.3699 [math-ph]] • R. Frassek, “ Q -operators, Yangian invariance and the quantum inverse scattering method,” arXiv:1412.3339 [hep-th].
- [4] V. G. Drinfeld, “A New realization of Yangians and quantized affine algebras,” *Sov. Math. Dokl.* **36** (1988) 212.
- [5] V. B. Kuznetsov, M. Salerno and E. K. Sklyanin, “Quantum Backlund transformation for the integrable DST model,” *J. Phys. A* **33** (2000) 171 [solv-int/9908002] • A. E. Kovalski and G. P. Pronko, “Baxter Q -operators for integrable DST chain,” arXiv:nlin/0203030.

- [6] V. V. Bazhanov and Z. Tsuboi, “Baxter’s Q -operators for supersymmetric spin chains,” Nucl. Phys. B **805** (2008) 451 [arXiv:0805.4274 [hep-th]] • Z. Tsuboi, “Asymptotic representations and q -oscillator solutions of the graded YangBaxter equation related to Baxter Q -operators,” Nucl. Phys. B **886** (2014) 1 [arXiv:1205.1471 [math-ph]] • N. Gromov, V. Kazakov and Z. Tsuboi, “ $PSU(2, 2|4)$ Character of Quasiclassical AdS/CFT,” JHEP **1007** (2010) 097 [arXiv:1002.3981 [hep-th]] • V. Kazakov, S. Leurent and Z. Tsuboi, “Baxter’s Q -operators and operatorial Backlund flow for quantum (super)-spin chains,” Commun. Math. Phys. **311** (2012) 787 [arXiv:1010.4022 [math-ph]].
- [7] D. Chicherin, S. Derkachov, D. Karakhanyan and R. Kirschner, “Baxter operators for arbitrary spin,” Nucl. Phys. B **854** (2012) 393 [arXiv:1106.4991 [hep-th]] • D. Chicherin, S. Derkachov, D. Karakhanyan and R. Kirschner, “Baxter operators for arbitrary spin II,” Nucl. Phys. B **854** (2012) 433 [arXiv:1107.0643 [hep-th]].
- [8] E. K. Sklyanin, “Quantum version of the method of inverse scattering problem,” J. Sov. Math. **19** (1982) 1546 [Zap. Nauchn. Semin. **95** (1980) 55].
- [9] J. T. Ding and I. Frenkel, “Isomorphism of two realizations of quantum affine algebra $U_q(\mathfrak{gl}(n))$,” Commun. Math. Phys. **156** (1993), 277.
- [10] J. Brundan and A. Kleshchev, “Parabolic Presentation of the Yangian $Y(\mathfrak{gl}(n))$,” Commun. Math. Phys. **254** (2005) 191 [arXiv:math/0407011].
- [11] A. Molev, “Yangians and classical Lie algebras,” Vol. 143, American Mathematical Soc., 1993 • A. Molev, M. Nazarov and G. Olshansky, “Yangians and classical Lie algebras,” Russ. Math. Surveys **51** (1996) 205 [hep-th/9409025] • N. J. MacKay, “Introduction to Yangian symmetry in integrable field theory,” Int. J. Mod. Phys. A **20** (2005) 7189 [hep-th/0409183] • A. Torrielli, “Yangians, S-matrices and AdS/CFT,” J. Phys. A **44** (2011) 263001 [arXiv:1104.2474 [hep-th]]
- [12] S. Z. Levendorskii, “On PBW bases for Yangians,” Lett. Math. Phys. **27** (1993) 37 • T. Matsumoto and A. Molev, “Representations of centrally extended Lie superalgebra $\mathfrak{psl}(2|2)$,” J. Math. Phys. **55** (2014) 091704 • T. Matsumoto, Poster presented at the conference on Integrability in Gauge and String Theory, 20-24 August 2012, ETH Zürich, <http://int12.itp.phys.ethz.ch/> • A. Torrielli, Talk delivered at the conference on Integrability in Gauge and String Theory, 20-24 August 2012, ETH Zürich, <http://int12.itp.phys.ethz.ch/>.